2023

MATHEMATICS — HONOURS

Paper : DSE-A(2)-2

(Mathematical Modelling)

Full Marks : 65

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Group - A

1. Answer ***all*** questions : 2×10

(To each question there are four alternatives of which one is correct. Choose the correct one with proper justification. One mark is for correct answer and one mark is for proper justification.)

(a) If $P_n(x)$ is Legendre's polynomial of degree n, then $\frac{d}{dx}\left(P_2(x)\right)$ at $x = 1$ will be

(i) 6 (ii) 3

(iii) 4 (iv) 1.

(b) In Monte-Carlo simulation of a fair die the random numbers x_n are generated between 0 and 3. Then which one denotes the event of obtaining 1 in dice?

(i) $0 < x_n < \frac{1}{6}$ (ii) $0 < x_n < 1$

(iii) $\frac{1}{3} < x_n < 1$ (iv) $0 < x_n < \frac{1}{2}$.

(c) Using linear congruence method $x_{n+1} = (9x_n + 11) \bmod 25$ with seed $x_0 = 11$, three random numbers generated are respectively

(i) 10, 11, 20 (ii) 11, 13, 19

(iii) 10, 1, 20 (iv) 11, 2, 20.

(d) In L.P.P. : $Z_{\max} = 3x + 4y$

subject to the condition $x - y \geq 0$

$-x + 3y \leq 3$

we have

(i) unique solution (ii) no feasible solution

(iii) infinitely many solution (iv) unbounded solution.

(e) The Laplace transform of $\frac{1}{\sqrt{\pi t}}$ is

(i) $\frac{1}{\sqrt{S}}$ (ii) $\frac{1}{\sqrt{\pi S}}$

(iii) $\frac{S}{\sqrt{\pi}}$ (iv) $\sqrt{\frac{S}{\pi}}$.

(f) Number of extreme points of the convex set

$$S = \left\{(x_1, x_2) \in \mathbb{R}^2;\ x_1 + x_2 \leq 2;\ -x_1 + 2x_2 \leq 2;\ x_1, x_2 \geq 0\right\}$$

is

(i) 2 (ii) 3

(iii) 4 (iv) 5.

(g) A shopkeeper handles only 1 person in 6 minutes while customer arrives in every 8 minutes. Then average queue length will be

(i) 3 persons (ii) 4 persons

(iii) 5 persons (iv) 6 persons.

(h) Cars arrive at a service station according to Poisson's distribution with mean rate 5 per hour and the service time is exponential with mean of 10 minutes. At steady state, average waiting time is

(i) 10 mins (ii) 20 mins

(iii) 25 mins (iv) 50 mins.

(i) The value of the limit $\lim_{x\to 0} \frac{J_n(x)}{x^n} (n \in N)$ is

(i) $\frac{1}{2^n}$ (ii) $\frac{n!}{2^n}$

(iii) $\frac{2^n}{n!}$ (iv) $\frac{1}{2^n n!}$.

(j) Inverse Laplace transform of $F(S) = \frac{S}{2S^2 - 8}$ is

(i) $\frac{\cos 2t}{2}$ (ii) $\frac{\cosh 2t}{2}$

(iii) $\frac{\sinh 2t}{2}$ (iv) $\frac{\sin 2t}{2}$.

Group - B

Unit - 1

2. Answer ***any two*** questions :

(a) (i) Show that $\left(1-2xz+z^2\right)^{-\frac{1}{2}} = \sum z^n P_n(x)$, where $P_n(x)$ is Legendre polynomial of degree n.

(ii) Prove that $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}\left(x^2-1\right)^n$ and using it find the value of $P_2(x)$. 5+5

(b) (i) Solve the equation $\dfrac{d^2y}{dt^2} - 3\dfrac{dy}{dt} + 2y = 1 - e^{2t}$, where $y = 1$, $\dfrac{dy}{dt} = 0$ at $t = 0$, using Laplace transform.

(ii) If $P_n(x)$ is Legendre polynomial of degree n, then show that $nP_n(x) = xP_n'(x) - P'_{n-1}(x)$, where prime denotes derivative w.r.t. x. 5+5

(c) (i) Show that $J_{-\frac{1}{2}}(x) = \sqrt{\dfrac{2}{\pi x}}\cos x$, where $J_n(x)$ is Bessel function of first kind of order 'n'.

(ii) Establish that $e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum t^n J_n(x)$, where $J_n(x)$ is Bessel function of first kind. 5+5

(d) (i) Find the series solution of the equation $2x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + \left(1-x^2\right)y = x^2$ about $x = 0$.

(ii) State convolution theorem of Laplace transform. Using it, find $L^{-1}\left[\dfrac{1}{p\left(p^2+4\right)^2}\right]$. 5+5

Unit - II

3. Answer ***any five*** questions :

(a) Solve the following L.P.P. by Graphical method : $Z_{\min} = -2x_1 + x_2$

$$\text{subject to } \begin{aligned} x_1 + x_2 &\geq 6 \\ 3x_1 + 2x_2 &\geq 16 \\ x_2 &\leq 9 \\ x_1,\ x_2 &\geq 0 \end{aligned}$$

If the objective function be changed to $2x_1 - x_2$, what will be minimum value of Z? 5

(b) Solve the following L.P.P. by algebraic method to get the optimal solution :

$$Z_{\min} = 3x + y$$

subject to $2x + 3y \le 6$

$x + y \ge 1$

$x, y \ge 0$ 5

(c) For the given L.P.P. : $Z_{\max} = x_1 + \frac{3}{2}x_2$

subject to $x_1 + x_2 \le 80$

$x_1 + 2x_2 \le 120$

$2x_1 + x_2 \le 140$

$x_1, x_2 \ge 0;$

find over what values of C_2, will the present solution be still optimal. 5

(d) In a railway marshalling yard, goods trains arrive at a rate 20 per day. Assuming inter-arrival time to follow exponential distribution and the service time is also exponential with an average of 54 minutes. Then determine the following :

(i) Average number of train in queue

(ii) Probability of queue size exceeding 10

(iii) Expected waiting time in queue

(iv) Average waiting time in system

(v) Estimate idle time of the yard. 5

(e) A small harbour has unloading facility for ships and only one ship can be unloaded at one time. Time span between arrivals of ships and unloading time of five ships are given below :

	S1	S2	S3	S4	S5
Time between ships (minutes)	20	30	15	120	25
Unloading time	55	45	60	75	80

Find the average time spent by the ships and total idle time of the harbour. 5

(f) Using Monte-Carlo simulation, write an algorithm to calculate the area between $y = x^2$ and $y = x$. 5

(g) Consider $x_0 = 9768$ as seed for middle square method and then find the sequence $\{x_n\}$ until obtained value is diminished to 0. Then consider $\{x_n \mid n = 0, 1, \ldots, 7\}$ as random numbers and divide them by 1000 to get $\{y_n \mid n = 0, 1, \ldots, 7\}$ random numbers between 0 and 1. Now if $0 < y_n \leq 0.5$ denotes event 'Head' and $0.5 < y_n \leq 1$ denotes event 'Tail', then find the probability of getting Head. 5

(h) A confectioner sells confectionery items. Past data of demand per week (in hundred kg) with frequency given below :

Demand/Week :	0	5	10	15	20	25
Frequency :	2	11	8	21	5	3

Using the following sequence of random numbers, generate demand for next 10 weeks. Also find the average demand per week.

[20, 72, 34, 54, 30, 22, 48, 74, 76, 02] 5